Universal Uncertainty Principle in the Measurement Operator Formalism

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Heisenberg's uncertainty principle has been understood to set a limitation on measurements; however, the long-standing mathematical formulation established by Heisenberg, Kennard, and Robertson does not allow such an interpretation. Recently, a new relation was found to give a universally valid relation between noise and disturbance in general quantum measurements, and it has become clear that the new relation plays a role of the first principle to derive various quantum limits on measurement and information processing in a unified treatment. This paper examines the above development on the noise-disturbance uncertainty principle in the model-independent approach based on the measurement operator formalism, which is widely accepted to describe a class of generalized measurements in the field of quantum information. We obtain explicit formulas for the noise and disturbance of measurements given by the measurement operators, and show that projective measurements do not satisfy the Heisenberg-type noise-disturbance relation that is typical in the gamma-ray microscope thought experiments. We also show that the disturbance on a Pauli operator of a projective measurement of another Pauli operator constantly equals $\sqrt{2}$, and examine how this measurement violates the Heisenberg-type relation but satisfies the new noise-disturbance relation.

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I. INTRODUCTION

Heisenberg's uncertainty principle has been understood to set a limitation on measurements by asserting a lower bound of the product of the imprecision of measuring one observable and the disturbance caused on another non-commuting observable. However, the long-standing mathematical formulation established by Heisenberg [1], Kennard [2], and Robertson [3] neither allows such an interpretation, nor has served to provide a reliable and general precision limit of measurements. In fact, it has been clarified through the controversy [4, 5, 6, 7, 8, 9, 10, 11] on the validity of the standard quantum limit for the gravitational wave detection [12, 13] that the purported reciprocal relation on noise and disturbance was not generally true [11, 14].

Although such a state of the art has undoubtedly resulted from the lack of reliable general measurement theory, the rapid development of the theory in the last two decades has made it possible to establish a universally valid uncertainty principle [15] for the most general class of quantum measurements, which will be useful for precision measurement and quantum information processing.

In Ref. [16] it was shown that the statistical properties of any physically possible quantum measurement is described by a normalized completely positive map-valued measure (CP instrument), and conversely that any CP instrument arises in this way. Thus, we naturally conclude that measurements are represented by CP instruments, just as states are represented by density operators and observables by self-adjoint operators. We have clarified the meaning of noise in the general measurement model and shown that this notion is equivalent to the distance of the probability operator-valued measure (POM) of the CP instrument from the observable to be measured, and hence the noise is independent of particular models but depends only on the POM of the instrument [17]. We have also shown that the disturbance in a given observable is determined only by the trace-preserving completely positive (TPCP) map associated with CP instrument [17].

Under the above formulation, we have generalized the Heisenberg-type noise-disturbance relation to a relation that holds for any measurements, from which conditions have been obtained for measurements to satisfy the original Heisenberg-type relation [15]. In particular, every measurement with the noise and the disturbance statistically independent from the measured object is proven to satisfy the Heisenberg-type relation [17].

In this paper, we shall examine the notions of noise and disturbance in the measurement operator formalism. The measurement operator formalism moderately generalizes the conventional projection operator approach to measurement and is often adopted in the field of quantum information [18]. In Section II, we discuss the uncertainty relation for standard deviations of non-commuting observables. This is the first rigorous formulation of Heisenberg's uncertainty principle. However, this formulation doe not directly mean the limitation on measurement of quantum objects typically described by the trade-off between noise and disturbance in the γ-ray microscope thought experiment [1]. In Section III, we discuss the uncertainty relation for joint measurements. The original form of this relation due to Arthurs and Kelly [19] shows that the product of the standard deviations of two meter-outputs of a position-momentum joint measurement has, if the measurement is jointly unbiased, the lower bound twice as large as that of the position and momentum. Obviously, this increase of the lower bound should be attributed to the additional noise imposed by the measuring interaction. We discuss the reformulation due to Ishikawa [20] and the present author [21] showing

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that the product of the root-mean-square noises imposed by the measuring interaction has the same lower bound as the product of the pre-measurement standard deviations of position and momentum. Thus, the Heisenberg's original formulation of the uncertainty principle on the limitation of measurement has been proved rigorously for jointly unbiased joint measurements. We argue that this relation also leads to the noise-disturbance relation but we need the unreasonable assumption that the disturbance be unbiased. In order to develop the theory of noise and disturbance of measurements in the model-independent formulation, in Section IV we introduce the concepts of CP instruments, POM, and CP maps to present the General Realization Theorem that mathematically determines the exact class of all the physically realizable measurements. Then, we introduce the measurement operator approach, which is widely accepted to describe a class of generalized measurements in the field of quantum information. We obtain explicit formulas for noise and disturbance of measurements given by the measurement operators. In Section V we show that projective measurements do not satisfy the Heisenberg-type noise-disturbance relation that is typical in the γ-ray microscope thought experiment. In Section VI we introduce the universal uncertainty principle for arbitrary measurements. From this, we give a general explicit criterion for measurements to satisfy the Heisenberg-type noise-disturbance relation. We also show that the disturbance on a Pauli operator of a projective measurement of another Pauli operator constantly equals to $\sqrt{2}$, and examine how this measurement violates the Heisenberg-type relation but satisfies the new noise-disturbance relation.

II. UNCERTAINTY PRINCIPLE WITHOUT MEASUREMENT THEORY

A. Heisenberg's uncertainty principle: The original formulation

In 1927 Heisenberg [1] proposed a reciprocal relation for measurement noise and disturbance by the famous γ ray microscope thought experiment. *Heisenberg's position-momentum uncertainty principle* can be expressed by

$$\Delta Q \Delta P \sim \hbar, \tag{1}$$

where ΔQ stands for the position measurement noise, "the mean error of Q", and ΔP stands for the momentum disturbance, "the discontinuous change of P".

Heisenberg claimed that the relation is a "straightforward mathematical consequence of the rule"

$$QP - PQ = i\hbar. \tag{2}$$

However, his proof did not fully account for the measurement noise or disturbance [22].

B. Kennard's relation: From noise to standard deviation

Immediately, Kennard [2] reformulated the relation as the famous inequality for the standard deviations of position and momentum. *Kennard's inequality* is given by

$$\sigma(Q)\,\sigma(P) \geq \frac{\hbar}{2}, \tag{3}$$

where σ stands for the standard deviation, i.e., $\sigma(X)^2 = (\langle X^2\rangle - \langle X\rangle^2)$ for any observable X.

C. Robertson's relation: From conjugate observables to any

Kennard's relation was soon generalized by Robertson [3] to arbitrary pairs of observables; see also [23]. *Robertson's inequality* is given by

$$\sigma(A)\,\sigma(B) \geq \frac{1}{2}|\langle[A,B]\rangle|, \tag{4}$$

where $\langle\cdots\rangle$ stands for the expectation value and $[A,B]$ stands for the commutator, i.e., $[A,B] = AB - BA$.

Robertson's proof bridges the uncertainty principle and the commutation relation simply appealing to the Schwarz inequality. However, Robertson's inequality wants a direct relevance to measurement noise nor disturbance, since the standard deviations depend only on the system's state but does not depend on the property of the measuring apparatus.

After presenting Robertson's proof, von Neumann [24, p. 237] wrote as follows. "With the foregoing considerations, we have comprehended only one phase of the uncertainty relations, that is, the formal one; for a complete understanding of these relations, it is still necessary to consider them from another point of view: from that of direct physical experience. For the uncertainty relations bear a more easily understandable and simpler relation to direct experience than many of the facts on which quantum mechanics was originally based, and therefore the above, entirely formal, derivation does not do them full justice."

Many text books have discuss a handful of thought experiments after formal derivation of Robertson's relation. However, for correct understanding of the uncertainty principle, we certainly need a reliable measurement theory rather than collecting more thought experiments.

III. UNCERTAINTY RELATIONS FOR JOINT MEASUREMENTS

A. Measuring processes

In order to obtain universally valid relations for measurement noise and disturbance, we should consider a

sufficiently general class of models of measurements. Generalizing von Neumann's description of measuring processes [24], we have introduced the following definition [16]: A *measuring process* for a quantum system $\mathbf{S}$ with state space (Hilbert space) $\mathcal{H}$ is a quadruple $\mathcal{M} = (\mathcal{K}, \rho_0, U, \{M_1, \ldots, M_n\})$ consisting of a Hilbert space $\mathcal{K}$ describing the state space of the probe $\mathbf{P}$, a state (density operator) ρ_0 on $\mathcal{K}$ describing the initial state of the probe, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ describing the time evolution of the composite system $\mathbf{S}+\mathbf{P}$ during the measuring interaction, and a set of mutually commuting observables $M_1, \ldots, M_n$ on $\mathcal{K}$ describing the probe observables to be detected in the state just after the measuring interaction.

A measuring process $\mathcal{M} = (\mathcal{K}, \rho_0, U, \{M_1, \ldots, M_n\})$ is called *pure* if ρ_0 is a pure state.

In the following, we shall deal with only the case $n = 1$ for simplicity of presentation; for the general definitions we refer the reader to Ref. [16]. The *output probability distribution* of measuring process $\mathcal{M} = (\mathcal{K}, \rho_0, U, M)$ on input state ρ is naturally defined by

$$\Pr\{\mathbf{x} \in \Delta \| \rho\} \ = \ \mathrm{Tr}_{\mathcal{K}}[\{I \otimes E^M(\Delta)\} U(\rho \otimes \rho_0) U^\dagger], \quad (5)$$

where $\mathbf{x}$ stands for the output of the measurement.

Let $\rho_{\{\mathbf{x}\in\Delta\}}$ be the state of $\mathbf{S}$ just after the measuring interaction given that the measurement leads to the output $\mathbf{x}$ in a Borel set Δ. Then, $\rho_{\{\mathbf{x}\in\Delta\}}$, the *output state on input ρ given* $\mathbf{x} \in \Delta$, is determined by

$$\rho_{\{\mathbf{x}\in\Delta\}} \ = \ \frac{\mathrm{Tr}_{\mathcal{K}}[\{I \otimes E^M(\Delta)\} U(\rho \otimes \rho_0) U^\dagger]}{\Pr\{\mathbf{x} \in \Delta \| \rho\}}, \quad (6)$$

provided that $\Pr\{\mathbf{x} \in \Delta \| \rho\} > 0$; otherwise $\rho_{\{\mathbf{x}\in\Delta\}}$ stands for an indefinite state. The state transformation $\rho \mapsto \rho_{\{\mathbf{x}\in\Delta\}}$ is called the *quantum state reduction* determined by the measuring process $\mathcal{M}$.

B. Arthurs and Kelly relation : From observables to meters

Suppose that the system $\mathbf{S}$ is a one-dimensional mass with position Q and momentum P. Let $\mathcal{M} = (\mathcal{K}, \rho_0, U, \{M_1, M_2\})$ be a measuring process with two meter observables M_1, M_2. Let $M_Q = U^\dagger(I \otimes M_1)U$ and $M_P = U^\dagger(I \otimes M_2)U$ the meter observables after the measuring interaction, the *posterior meters*. Then, we say that $\mathcal{M}$ is a *jointly unbiased position-momentum joint measurement* if

$$\mathrm{Tr}[M_Q(\rho \otimes \rho_0)] \ = \ \mathrm{Tr}[Q\rho], \quad (7)$$
$$\mathrm{Tr}[M_P(\rho \otimes \rho_0)] \ = \ \mathrm{Tr}[P\rho] \quad (8)$$

for any $\rho \in \mathcal{S}(\mathcal{H})$ with $\mathrm{Tr}[Q^2\rho], \mathrm{Tr}[P^2\rho] < \infty$, where $\mathcal{S}(\mathcal{H})$ stands for the space of density operators on $\mathcal{H}$. Then, Arthurs and Kelley [19] showed that every jointly unbiased position-momentum joint measurement satisfies

$$\sigma(M_Q)\, \sigma(M_P) \geq \hbar. \quad (9)$$

If we measure Q and P separately with ideal measuring apparatuses for many samples prepared in the same state, then $\sigma(Q)$ and $\sigma(P)$ can be considered as the standard deviations of the output of each measurement. However, $\sigma(Q)$ and $\sigma(P)$ can by no means be considered as the standard deviations of the outputs of a joint measurement carried out by a single apparatus.

C. Arthurs and Goodman relation : Meter uncertainty relation

The relation (9) has been generalized to arbitrary pairs of observables as follows. Let $\mathcal{M} = (\mathcal{K}, \rho_0, U, \{M_1, M_2\})$ be a measuring process with two meter observables M_1, M_2. Let A, B be two observables of $\mathbf{S}$. Let $M_A = U^\dagger(I \otimes M_1)U$ and $M_B = U^\dagger(I \otimes M_2)U$ be the posterior meters. Then, we say that $\mathcal{M}$ is a *jointly unbiased joint measurement of the pair* (A, B) if

$$\mathrm{Tr}[M_A(\rho \otimes \rho_0)] \ = \ \mathrm{Tr}[A\rho], \quad (10)$$
$$\mathrm{Tr}[M_B(\rho \otimes \rho_0)] \ = \ \mathrm{Tr}[B\rho] \quad (11)$$

for any $\rho \in \mathcal{S}(\mathcal{H})$ with $\mathrm{Tr}[A^2\rho], \mathrm{Tr}[B^2\rho] < \infty$. Then, Arthurs and Goodman [25] showed that any jointly unbiased joint measurement of pair (A, B) satisfies

$$\sigma(M_A)\, \sigma(M_B) \geq |\langle [A, B] \rangle|. \quad (12)$$

D. Ishikawa and Ozawa: From meter uncertainty to measurement noise

Thus, the meter uncertainty product $\sigma(M_A)\, \sigma(M_B)$ has the lower bound twice as large as the observable uncertainty product $\sigma(A)\, \sigma(B)$. This, increase of the uncertainty product can be considered to be yielded by the intrinsic noise from the measuring process other than the initial deviation. In order to quantify the above intrinsic noise, we introduce *noise operators* N_A, N_B defined by

$$N_A \ = \ M_A - A \otimes I, \quad (13)$$
$$N_B \ = \ M_B - B \otimes I. \quad (14)$$

Then, the measurement is jointly unbiased if and only if $\langle N_A \rangle = \langle N_B \rangle = 0$.

Then, Ishikawa [20] and Ozawa [21] showed that any jointly unbiased joint measurement of pair (A, B) satisfies

$$\sigma(N_A)\, \sigma(N_B) \geq \frac{1}{2} |\langle [A, B] \rangle|. \quad (15)$$

E. Uncertainty principle for jointly unbiased joint measurements

The root-mean-square noises are naturally defined as the root-mean-square of the noise operator, i.e.,

$$\epsilon(A) = \langle N_A^2 \rangle^{1/2}, \quad (16)$$
$$\epsilon(B) = \langle N_B^2 \rangle^{1/2}. \quad (17)$$

Then, we have $\epsilon(A) \ge \sigma(N_A)$ and $\epsilon(B) \ge \sigma(N_B)$, so that we can conclude that the Heisenberg-type joint noise relation

$$\epsilon(A)\epsilon(B) \ge \frac{1}{2}|\langle[A,B]\rangle| \tag{18}$$

holds for any jointly unbiased joint measurement of pair (A,B) [20, 21].

F. From joint noise relation to noise-disturbance relation

The question why joint measurements have the inevitable noise may be answered by the notion of disturbance caused by measurements. The above considerations can be applied to obtain the relation between measurement noise and disturbance as follows.

Let $(\mathcal{K}, \rho_0, U, M)$ be a measuring process for the system $\mathbf{S}$. Let A, B be two observables on $\mathcal{H}$. The *noise operator* N_A and the *disturbance operator* D_B are defined by

$$\begin{aligned} N_A &= U^\dagger(I\otimes M)U - A\otimes I, & (19)\\ D_B &= U^\dagger(B\otimes I)U - B\otimes I. & (20)\end{aligned}$$

The noise operator N_A represents the noise in measuring A. The disturbance operator D_B represents the disturbance caused on B during the measuring interaction. We naturally define the *root-mean-square noise* $\epsilon(A)$ and the *root-mean-square disturbance* $\eta(B)$ by

$$\begin{aligned} \epsilon(A) &= \langle N_A^2\rangle^{1/2}, & (21)\\ \eta(B) &= \langle D_B^2\rangle^{1/2}. & (22)\end{aligned}$$

We say that the measurement is an *unbiased measurement* of A if $\mathrm{Tr}[N_A\rho] = 0$ for all $\rho \in \mathcal{S}(\mathcal{H})$, and the measurement has *unbiased disturbance* if $\mathrm{Tr}[D_B\rho] = 0$ for all $\rho \in \mathcal{S}(\mathcal{H})$. Then, by the same mathematics as above, we can show that unbiased measurements of A with unbiased disturbance on B satisfy [15]

$$\sigma(N_A)\,\sigma(D_B) \ge \frac{1}{2}|\langle[A,B]\rangle|. \tag{23}$$

Thus, unbiased measurements of A with unbiased disturbance on B satisfy the the *Heisenberg-type noise-disturbance relation for* (A,B) [15]

$$\epsilon(A)\eta(B) \ge \frac{1}{2}|\langle[A,B]\rangle|. \tag{24}$$

From the above, it is tempting to call an unbiased measurement with unbiased disturbance a good measurement and to state that every good measurement satisfies the Heisenberg-type noise-disturbance relation. However, this cannot be justified as shown in the later sections.

IV. MODEL-INDEPENDENT APPROACH TO UNCERTAINTY PRINCIPLE

A. Completely positive instruments

In order to describe the statistical properties of measuring processes by a unified mathematical object, we introduce some mathematical definitions.

A bounded linear transformation T on the space $\tau c(\mathcal{H})$ of trace class operators on $\mathcal{H}$ is called a completely positive (CP) map if the trivial extension $T\otimes id$ to $\tau c(\mathcal{H}\otimes\mathcal{H})$ is positive. Every CP map T on $\tau c(\mathcal{H})$ has a family $\{\Lambda_j\}$ of bounded operators, called *Kraus operators* for T, such that $T\rho = \sum_j \Lambda_j\rho\Lambda_j^\dagger$ for all $\rho \in \tau c(\mathcal{H})$ [26, 27], and the converse is obviously true.

A mapping $\mathcal{I}$ from each Borel set Δ in the real line $\mathbf{R}$ to a bounded linear transformation $\mathcal{I}(\Delta)$ on $\tau c(\mathcal{H})$ is called a *CP instrument* if it satisfies the following conditions [16].

(i) (Complete positivity) The linear transformation $\mathcal{I}(\Delta)$ is completely positive for any Borel set Δ.

(ii) (Countable additivity) For any disjoint sequence of Borel sets Δ_j, we have

$$\mathcal{I}(\bigcup_j \Delta_j)\rho = \sum_j \mathcal{I}(\Delta_j)\rho. \tag{25}$$

(iii) (Unity of total probability) For any density operator ρ,

$$\mathrm{Tr}[\mathcal{I}(\mathbf{R})\rho] = 1. \tag{26}$$

Let $\mathcal{M} = (\mathcal{K}, \rho_0, U, M)$ be a measuring process. The relation

$$\mathcal{I}(\Delta)\rho = \mathrm{Tr}_{\mathcal{K}}[\{I\otimes E^M(\Delta)\}U(\rho\otimes\rho_0)U^\dagger] \tag{27}$$

defines a CP instrument, called the *instrument determined by the measuring process* $\mathcal{M} = (\mathcal{K}, \rho_0, U, M)$. Then, from Eq. (5) and Eq. (6) the instrument $\mathcal{I}$ represents both the output probability distribution and the quantum state reduction by

$$\begin{aligned} \Pr\{\mathbf{x}\in\Delta\|\rho\} &= \mathrm{Tr}[\mathcal{I}(\Delta)\rho], & (28)\\ \rho_{\{\mathbf{x}\in\Delta\}} &= \frac{\mathcal{I}(\Delta)\rho}{\mathrm{Tr}[\mathcal{I}(\Delta)\rho]}, & (29)\end{aligned}$$

provided that $\mathrm{Tr}[\mathcal{I}(\Delta)\rho] > 0$ [16].

Given a CP instrument $\mathcal{I}$ and a Borel set Δ, the *dual CP map* $\mathcal{I}(\Delta)^*$ on the space $\mathcal{L}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ is defined by

$$\mathrm{Tr}[\{\mathcal{I}(\Delta)^*A\}\rho] = \mathrm{Tr}[A\{\mathcal{I}(\Delta)\rho\}] \tag{30}$$

for any $A \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$. If $\{\Lambda_j\}$ is Kraus operators of $\mathcal{I}(\Delta)$, i.e, $\mathcal{I}(\Delta)\rho = \sum_j \Lambda_j\rho\Lambda_j^\dagger$, we have $\mathcal{I}(\Delta)^*A = \sum_j \Lambda_j^\dagger A\Lambda_j$. Then, it is easy to see that for any instrument $\mathcal{I}$, the relation

$$\Pi(\Delta) = \mathcal{I}(\Delta)^*I \tag{31}$$

for all Borel set Δ defines a unique POM, *the POM determined by instrument* $\mathcal{I}$.

Let $\mathcal{I}$ be the instrument determined by measuring process $\mathcal{M}$. Then, the POM determined by the instrument $\mathcal{I}$ satisfies

$$\Pi(\Delta) = \mathrm{Tr}_{\mathcal{K}}[U^\dagger\{I \otimes E^M(\Delta)\}U(I \otimes \rho_0)], \qquad (32)$$

and we have the *generalized Born statistical formula*

$$\Pr\{\mathbf{x} \in \Delta \| \rho\} \;=\; \mathrm{Tr}[\Pi(\Delta)\rho]. \qquad (33)$$

For the case $\Delta = \mathbf{R}$, the state transformation $T : \rho \mapsto \rho_{\{\mathbf{x}\in\mathbf{R}\}}$ is called *the nonselective state reduction*. From Eq. (29), the nonselective state reduction T is a trace-preserving completely positive (TPCP) map determined by

$$T = \mathcal{I}(\mathbf{R}), \qquad (34)$$

and we have

$$T\rho \;=\; \mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes \rho_0)U^\dagger], \qquad (35)$$
$$T^*A \;=\; \mathrm{Tr}_{\mathcal{K}}[U^\dagger(A \otimes I)U(I \otimes \rho_0)] \qquad (36)$$

for all $A \in \mathcal{L}(\mathcal{H})$ and $\rho \in \tau c(\mathcal{H})$.

B. General realization theorem

In the preceding subsections, we have shown that any measuring process determines a CP instrument. Then, it is natural to ask whether the notion of measuring process is too restrictive or whether the notion of CP instrument is too general. The following theorem shows that the notion of measuring process is general enough and the notion of CP instruments characterizes all the possible measurements [16, 28].

Theorem 1. (General Realization Theorem) *For every completely positive instrument* $\mathcal{I}$*, there is a pure measuring process* $\mathcal{M} = (\mathcal{K}, |\xi\rangle\langle\xi|, U, M)$ *such that* $\mathcal{I}$ *is determined by* $\mathcal{M}$.

Before the above theorem was found, there had been many proposals for mathematical description of measurements, but the theorem definitely determined which proposals are consistent with quantum mechanics and which are not [14].

The General Realization Theorem has the following corollaries [16].

Corollary 2. (Realization of POMs) *Every POM* Π *can be represented as*

$$\Pi(\Delta) = \mathrm{Tr}_{\mathcal{K}}[U^\dagger\{I \otimes E^M(\Delta)\}U(I \otimes |\xi\rangle\langle\xi|)]. \qquad (37)$$

Corollary 3. (Realization of TPCP maps) *Every TPCP map* T *can be represented as*

$$T\rho \;=\; \mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger], \qquad (38)$$
$$T^*A \;=\; \mathrm{Tr}_{\mathcal{K}}[U^\dagger(A \otimes I)U(I \otimes |\xi\rangle\langle\xi|)]. \qquad (39)$$

Corollary 2 follows from the General Realization Theorem applied to a CP instrument such that $\mathcal{I}(\Delta)^*I = \Pi(\Delta)$; a trivial example is given by $\mathcal{I}(\Delta)\rho = \mathrm{Tr}[\Pi(\Delta)\rho]\rho_0$ where ρ_0 is a fixed state. Corollary 3 follows from the General Realization Theorem applied to a CP instrument such that $\mathcal{I}(\mathbf{R}) = T$; a trivial example is given by $\mathcal{I}(\Delta)\rho = \mu(\Delta)T(\rho)$ where μ is a fixed probability measure. An equivalent form of Corollary 3 was found by Kraus [27].

C. Measurement operator formalism

In the field of quantum information, a particular description of measurements is commonly adopted [18]. A family $\{M_m\}$ of operators with one real parameter m is called a family of *measurement operators* if

$$\sum_m M_m^\dagger M_m = I \qquad (40)$$

and is supposed to describe a measurement such that

$$\Pr\{\mathbf{x} = m \| \rho\} \;=\; \mathrm{Tr}[M_m^\dagger M_m \rho], \qquad (41)$$
$$\rho_{\{\mathbf{x}=m\}} \;=\; \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}[M_m^\dagger M_m \rho]} \qquad (42)$$

for all ρ.

It is easy to judge whether this proposed description of measurement is consistent or not, in the light of the General Realization Theorem as follows. It is easy to see that the relation

$$\mathcal{I}(\Delta)\rho = \sum_{m\in\Delta} M_m \rho M_m^\dagger \qquad (43)$$

defines a CP instrument; complete positivity of $\mathcal{I}(\Delta)$ follows from the fact that $\{M_m\}_{m\in\Delta}$ is a family of Kraus operators of $\mathcal{I}(\Delta)$, countable additivity follows from the property of summation, and unity of probability follows from Eq. (40). Thus, by the General Realization Theorem, we have a measuring process $\mathcal{M} = (\mathcal{K}, |\xi\rangle\langle\xi|, U, M)$ such that

$$M_m \rho M_m^\dagger = \mathrm{Tr}_{\mathcal{K}}[(I \otimes E_m^M)U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger], \qquad (44)$$

where $E_m^M = E^M(\{m\})$, i.e., $M = \sum_m m E_m^M$, so that we have

$$\Pr\{\mathbf{x} = m \| \rho\} \;=\; \mathrm{Tr}[(I \otimes E_m^M)U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger], \qquad (45)$$
$$\rho_{\{\mathbf{x}=m\}} \;=\; \frac{\mathrm{Tr}_{\mathcal{K}}[(I \otimes E_m^M)U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger]}{\mathrm{Tr}[(I \otimes E_m^M)U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger]}. \qquad (46)$$

The POM of this measurement is given by

$$\Pi(\Delta) \;=\; \sum_{m\in\Delta} \Pi_m, \qquad (47)$$
$$\Pi_m \;=\; M_m^\dagger M_m, \qquad (48)$$

where $\Delta \subset \mathbf{R}$. For any state vector ψ, we have

$$
\begin{aligned}
\langle\psi|\Pi_m|\psi\rangle &= \langle\psi|M_m^\dagger M_m|\psi\rangle \\
&= \mathrm{Tr}[M_m|\psi\rangle\langle\psi|M_m^\dagger] \\
&= \mathrm{Tr}[U^\dagger(I\otimes E_m^M)U(|\psi\rangle\langle\psi|\otimes|\xi\rangle\langle\xi|)] \\
&= \langle\psi|\langle\xi|U^\dagger(I\otimes E_m^M)U|\xi\rangle_{\mathcal{K}}|\psi\rangle,
\end{aligned}
$$

where $\langle\cdots|\cdots\rangle_{\mathcal{K}}$ stands for the partial inner product over $\mathcal{K}$. Thus, we have

$$
\Pi_m = \langle\xi|U^\dagger(I\otimes E_m^M)U|\xi\rangle_{\mathcal{K}}. \tag{49}
$$

The TPCP map T describing the nonselective state reduction and its dual map T^* are given by

$$
\begin{aligned}
T\rho &= \sum_m M_m\rho M_m^\dagger, &(50)\\
T^*A &= \sum_m M_m^\dagger A M_m. &(51)
\end{aligned}
$$

D. Noise of POMs

For any pure measuring process $\mathcal{M}=(\mathcal{K},|\xi\rangle\langle\xi|,U,M)$, the root-mean-square noise $\epsilon(A)$ of $\mathcal{M}$ for measuring A on input ψ is given by

$$
\epsilon(A)^2 = \langle\psi\otimes\xi|(M'-A\otimes I)^2|\psi\otimes\xi\rangle, \tag{52}
$$

where $M'=U^\dagger(I\otimes M)U$. It is easy to rewrite the above formula as [17]

$$
\begin{aligned}
\epsilon(A)^2 &= \langle\psi|A^2|\psi\rangle + \langle\psi\otimes\xi|(M')^2|\psi\otimes\xi\rangle \\
&\quad +\langle\psi\otimes\xi|M'|\psi\otimes\xi\rangle + \langle A\psi\otimes\xi|M'|A\psi\otimes\xi\rangle \\
&\quad -\langle(A+I)\psi\otimes\xi|M'|(A+I)\psi\otimes\xi\rangle. &(53)
\end{aligned}
$$

Thus, the root-mean-square noise of the measurement is determined by the second and the first moments of the output probability distribution on input ψ, the first moments of the output probability distributions on inputs $A\psi$ and $(A+I)\psi$; here, we omit obvious normalization factors of state vectors. Since those quantities are determined only by the output probability distributions on several input states, so that it is clear that the root-mean-square noise is determined solely by the POM Π of the measuring process.

In Ref. [29] the case $\epsilon(A)=0$ is thoroughly studied comparing with the notion of precise measurement that is characterized by the condition that the posterior meter M' and the measured observable A are perfectly correlated in the initial state $\psi\otimes\xi$, and it is shown that the measurement is precise on input ψ if and only if $\epsilon(A)=0$ for all states $A^n\psi$ with $n=1,2,\dots$.

For a one-parameter family of measurement operators, $\{M_m\}$, through a realization given by Eq. (44) we have

$$
\begin{aligned}
\langle N_A^2\rangle &= \langle\psi\otimes\xi|(U^\dagger(I\otimes M)U-A\otimes I)^2|\psi\otimes\xi\rangle \\
&= \sum_m m^2\langle\psi\otimes\xi|E_m^{M'}|\psi\otimes\xi\rangle - m\langle\psi\otimes\xi|E_m^{M'}|A\psi\otimes\xi\rangle - m\langle A\psi\otimes\xi|E_m^{M'}|\psi\otimes\xi\rangle + \langle\psi\otimes\xi|A^2|\psi\otimes\xi\rangle \\
&= \sum_m m^2\langle\psi|\Pi_m|\psi\rangle - m\langle\psi|\Pi_m|A\psi\rangle - m\langle A\psi|\Pi_m|\psi\rangle + \langle\psi|A^2|\psi\rangle \\
&= \sum_m \langle\psi|m^2\Pi_m - m\Pi_m A - mA\Pi_m + A\Pi_m A|\psi\rangle \\
&= \sum_m \langle\psi|m^2 M_m^\dagger M_m - mM_m^\dagger M_m A - mAM_m^\dagger M_m + AM_m^\dagger M_m A|\psi\rangle \\
&= \sum_m \langle\psi|(mM_m - M_m A)^\dagger(mM_m - M_m A)|\psi\rangle \\
&= \sum_m \|M_m(m-A)\psi\|^2,
\end{aligned}
$$

and we have

$$
\epsilon(A) = (\sum_m \|M_m(m-A)\psi\|^2)^{1/2}. \tag{54}
$$

Note that if the ranges of measurement operators M_m are mutually orthogonal, by the Pythagoras theorem we have

$$
\sum_m \|M_m(m-A)\psi\|^2 = \|(\sum_m mM_m - A)\psi\|^2, \tag{55}
$$

and hence we have

$$
\epsilon(A) = \|\sum_m mM_m\psi - A\psi\|. \tag{56}
$$

E. Disturbance of TPCP maps

For any pure measuring process $\mathcal{M} = (\mathcal{K}, |\xi\rangle\langle\xi|, U, M)$, it is easy to check that the root-mean-square disturbance $\eta(B)$ of $\mathcal{M}$ caused in a bounded observable B on input ψ is given by [17]

$$\eta(B)^2 \;=\; \langle\psi|T^*(B^2) - BT^*(B) - T^*(B)B + B^2|\psi\rangle, \quad (57)$$

where T^* is the dual CP map determined by Eq. (36) with $\rho_0 = |\xi\rangle\langle\xi|$. Thus, the root-mean-square disturbance $\eta(B)$ is determined by the TPCP map $T = \mathcal{I}(\mathbf{R})$ determined by the measuring process $\mathcal{M}$.

Let $\{M_m\}$ be a family of measurement operators and let $T\rho = \sum_m M_m\rho M_m^\dagger$ the corresponding TPCP map. We have also

$$\begin{aligned}
\langle D_B^2\rangle &= \langle\psi\otimes\xi|(U^\dagger(B\otimes I)U - B\otimes I)^2|\psi\otimes\xi\rangle\\
&= \langle\psi\otimes\xi|U^\dagger(B\otimes I)^2U|\psi\otimes\xi\rangle - \langle\psi\otimes\xi|U^\dagger(B\otimes I)U|B\psi\otimes\xi\rangle - \langle B\psi\otimes\xi|U^\dagger(B\otimes I)U|\psi\otimes\xi\rangle + \langle\psi|B^2|\psi\rangle\\
&= \langle\psi|T(B^2)|\psi\rangle - \langle\psi|T(B)|B\psi\rangle - \langle B\psi|T(B)|\psi\rangle + \langle\psi|B^2|\psi\rangle\\
&= \langle\psi|\sum_m M_m^\dagger B^2 M_m - \sum_m M_m^\dagger BM_mB - \sum_m BM_m^\dagger BM_m + \sum_m BM_m^\dagger M_mB|\psi\rangle\\
&= \sum_m\langle\psi|M_m^\dagger B^2M_m - M_m^\dagger BM_mB - BM_m^\dagger BM_m + BM_m^\dagger M_mB|\psi\rangle\\
&= \sum_m\langle\psi|[M_m,B]^\dagger[M_m,B]|\psi\rangle\\
&= \sum_m\|[M_m,B]\psi\|^2.
\end{aligned}$$

Thus, we have

$$\eta(B) \;=\; (\sum_m\|[M_m,B]\psi\|^2)^{1/2}. \quad (58)$$

F. Model-independent approach to joint measurements

To apply the results on joint measurements to the measurement operator formalism, we consider two-parameter family of measurement operators, $\{M_{a,b}\}$, satisfying

$$\sum_{a,b} M_{a,b}^\dagger M_{a,b} = I, \quad (59)$$

with describes a measurement such that

$$\begin{aligned}
\Pr\{\mathbf{x}=a,\mathbf{y}=b\|\psi\} &= \|M_{a,b}\psi\|^2, &(60)\\
\psi_{\{\mathbf{x}=a,\mathbf{y}=b\}} &= \frac{M_{a,b}\psi}{\|M_{a,b}\psi\|} &(61)
\end{aligned}$$

for any (pure) state ψ. Then, by the General Realization Theorem, we have a measuring process $\mathcal{M} = (\mathcal{K}, |\xi\rangle\langle\xi|, U, \{M_1, M_2\})$ such that

$$\begin{aligned}
&M_{a,b}|\psi\rangle\langle\psi|M_{a,b}^\dagger\\
&\quad= \mathrm{Tr}_{\mathcal{K}}[(I\otimes E_a^{M_1}E_b^{M_2})U(I\otimes|\xi\rangle\langle\xi|)U^\dagger]. \quad (62)
\end{aligned}$$

The instrument and the POM of this measurement are given by

$$\begin{aligned}
\mathcal{I}(\Delta)\rho &= \sum_{(a,b)\in\Delta} M_{a,b}\rho M_{a,b}^\dagger, &(63)\\
\Pi(\Delta) &= \sum_{(a,b)\in\Delta}\Pi_{a,b}, &(64)\\
\Pi_{a,b} &= M_{a,b}^\dagger M_{a,b}, &(65)
\end{aligned}$$

where $\Delta\subset\mathbf{R}^2$. We define the marginal POMs Π^A and Π^B by

$$\begin{aligned}
\Pi^A(\Delta) &= \sum_{a\in\Delta}\Pi_a^A,\\
\Pi_a^A &= \sum_b M_{a,b}^\dagger M_{a,b},\\
\Pi^B(\Delta) &= \sum_{b\in\Delta}\Pi_b^B,\\
\Pi_b^B &= \sum_a M_{a,b}^\dagger M_{a,b},
\end{aligned}$$

where $\Delta\subset\mathbf{R}$. For any state vector ψ, we have

$$\begin{aligned}
\langle\psi|\Pi_a^A|\psi\rangle &= \sum_b \mathrm{Tr}[(I\otimes E_a^{M_1}E_b^{M_2})U(|\psi\rangle\langle\psi|\otimes|\xi\rangle\langle\xi|)U^\dagger]\\
&= \mathrm{Tr}[U^\dagger(I\otimes E_a^{M_1})U(|\psi\rangle\langle\psi|\otimes|\xi\rangle\langle\xi|)]\\
&= \langle\psi|\langle\xi|E_a^{M_A}|\xi\rangle_{\mathcal{K}}|\psi\rangle,
\end{aligned}$$

and the analogous relation also holds for Π_b^B. Thus, we have

$$\Pi_a^A = \langle\xi|E_a^{M_A}|\xi\rangle_{\mathcal{K}}, \quad (66)$$
$$\Pi_b^B = \langle\xi|E_b^{M_B}|\xi\rangle_{\mathcal{K}}. \quad (67)$$

If $\rho = |\psi\rangle\langle\psi|$ and $\rho_0 = |\xi\rangle\langle\xi|$, we have

$$\begin{aligned}\mathrm{Tr}[M_A(\rho\otimes\rho_0)] &= \sum_a a\langle\psi|\langle\xi|E_a^{M_A}|\xi\rangle_{\mathcal{K}}|\psi\rangle \\ &= \langle\psi|\sum_a a\Pi_a^A|\psi\rangle \\ &= \mathrm{Tr}[\sum_a a\Pi_a^A\rho]\end{aligned}$$

and analogously we have

$$\mathrm{Tr}[M_B(\rho\otimes\rho_0)] = \mathrm{Tr}[\sum_b b\Pi_b^B\rho].$$

It follows that $\mathcal{M}$ is a jointly unbiased joint measurement of (A, B) if and only if $\sum_a a\Pi_a^A = A$ and $\sum_b b\Pi_a^B = B$. In this case, we have $\langle M_A\rangle = \langle A\rangle$ and $\langle M_B\rangle = \langle B\rangle$, and

$$\begin{aligned}\sigma(M_A)^2 &= \sum_a (a - \langle M_A\rangle)^2\langle\psi\otimes\xi|E_a^{M_A}|\psi\otimes\xi\rangle \\ &= \sum_a (a-\langle A\rangle)^2\langle\psi|\Pi_a^A|\psi\rangle \\ &= \sum_a \langle\psi|(a-\langle A\rangle)^2\Pi_a^A|\psi\rangle \\ &= \sum_a \langle\psi|(a-\langle A\rangle)^2\sum_b M_{a,b}^\dagger M_{a,b}|\psi\rangle \\ &= \sum_{a,b}\|(a-\langle A\rangle)M_{a,b}\psi\|^2.\end{aligned}$$

The analogous relation also holds for M_B, and hence we have

$$\sigma(M_A) = (\sum_{a,b}\|(a-\langle A\rangle)M_{a,b}\psi\|^2)^{1/2}, \quad (68)$$
$$\sigma(M_B) = (\sum_{a,b}\|(b-\langle B\rangle)M_{a,b}\psi\|^2)^{1/2}. \quad (69)$$

By calculations similar to what lead to Eq. (54), we have

$$\langle N_A^2\rangle = \sum_{a,b}\|M_{a,b}(a-A)\psi\|^2.$$

The analogous relation also holds for M_B. Since $\sigma(N_A)^2 = \langle N_A^2\rangle - \langle N_A\rangle^2 = \langle N_A^2\rangle$, we have

$$\sigma(N_A) = (\sum_{a,b}\|M_{a,b}(a-A)\psi\|^2)^{1/2}, \quad (70)$$
$$\sigma(N_B) = (\sum_{a,b}\|M_{a,b}(b-B)\psi\|^2)^{1/2}. \quad (71)$$

We summarize the uncertainty relation for jointly unbiased joint measurement in the measurement operator formalism.

Theorem 4. *Let $\{M_{a,b}\}$ be a two-parameter family of measurement operators. Let $A = \sum_{a,b} aM_{a,b}^\dagger M_{a,b}$ and $B = \sum_{a,b} bM_{a,b}^\dagger M_{a,b}$. Then, we have*

$$\epsilon(A)\epsilon(B) \ge \frac{1}{2}|\langle[A,B]\rangle|, \quad (72)$$

and

$$\epsilon(A) = (\sum_{a,b}\|M_{a,b}(a-A)\psi\|^2)^{1/2}, \quad (73)$$
$$\epsilon(B) = (\sum_{a,b}\|M_{a,b}(b-B)\psi\|^2)^{1/2}. \quad (74)$$

V. PROJECTIVE MEASUREMENTS DO NOT OBEY THE HEISENBERG-TYPE NOISE-DISTURBANCE RELATION

One of the most typical class of good measurements is the projective measurements defined as follows. A measurement with instrument $\mathcal{I}$ is called the *projective measurement* of a discrete observable A with spectral decomposition $A = \sum_m mE_m^A$ if

$$\mathcal{I}(\{m\})\rho = E_m^A\rho E_m^A. \quad (75)$$

Thus, the projective measurement of A is a measurement with measurement operators $\{M_m\} = \{E_m^A\}$, and we have

$$\sum_m mM_m = A. \quad (76)$$

Now we shall show the following

Theorem 5. *No projective measurements satisfy the Heisenberg-type noise-disturbance relation for (A, B) if B is bounded and $\langle[A,B]\rangle \ne 0$.*

The proof runs as follows. First, we note that the projective measurement of A is a noiseless measurement of A. In fact, from Eq. (56) and Eq. (76), we have

$$\epsilon(A) = \|\sum_m mM_m\psi - A\psi\| = 0. \quad (77)$$

On the other hand, we can show the following.

Lemma 6. *The disturbance of a bounded operator B caused by any TPCP map T is at most $2\|B\|$, i.e.,*

$$\eta(B) \le 2\|B\|. \quad (78)$$

To prove the above lemma, we can assume without any loss of generality that $\{M_m\}$ is a family of Kraus

operators of T. From Eq. (58) we have

$$\begin{aligned}
\eta(B)^2 &= \sum_m \|[M_m, B]\psi\|^2 \\
&= \sum_m \|M_m B\psi - BM_m\psi\|^2 \\
&\le \sum_m 2\|M_m B\psi\|^2 + 2\|BM_m\psi\|^2 \\
&\le 2\sum_m \|M_m B\psi\|^2 + 2\|B\|^2 \sum_m \|M_m\psi\|^2 \\
&\le 2\|B\psi\|^2 + 2\|B\|^2\|\psi\|^2 \\
&\le 4\|B\|^2.
\end{aligned}$$

Thus, we have proved Eq. (78).

From the above argument, we conclude that the projective measurement of A satisfies

$$\epsilon(A)\eta(B) = 0 \tag{79}$$

for any bounded observable B, so that the projective measurement of A do not satisfy the Heisenberg-type noise-disturbance relation for the noise in A measurement and the disturbance of B, provided that B is bounded and $\langle [A, B]\rangle \neq 0$.

If the projective measurement were to have unbiased disturbance, then it should satisfy the Heisenberg-type noise-disturbance relation. Thus, we can also conclude that *the projective measurement of any (discrete) observable A has no unbiased disturbance on a bounded observable B with $\langle [A, B]\rangle \neq 0$.*

VI. UNIVERSALLY VALID REFORMULATION OF UNCERTAINTY PRINCIPLE

A. Universal uncertainty principle

We have argued that the Heisenberg-type noise-disturbance relation is often unreliable. Recently, the present author [15] proposed a new relation for noise and disturbance with a rigorous proof of the universal validity.

Theorem 7. (Universal Uncertainty Principle) *For any measuring process $\mathcal{M} = (\mathcal{K}, \rho_0, U, M)$ and observables A, B, we have*

$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \ge \frac{1}{2}|\langle [A, B]\rangle|. \tag{80}$$

A dimensionless form of the universal uncertainty relation is given by [30]

$$\frac{\epsilon(A)\eta(B)}{\sigma(A)\sigma(B)} + \frac{\epsilon(A)}{\sigma(A)} + \frac{\eta(B)}{\sigma(B)} \ge \frac{|\langle [A, B]\rangle|}{2\sigma(A)\sigma(B)}. \tag{81}$$

For further accounts on the universal uncertainty principle, including foundations and applications, we refer the reader to [17, 22, 31, 32, 33].

B. When the Heisenberg-type noise-disturbance relation holds?

We introduce the *mean noise operator* and the *mean disturbance operator* of the measuring process $\mathcal{M} = (\mathcal{K}, \rho_0, U, M)$ by

$$\begin{aligned}
n_A &= \mathrm{Tr}_{\mathcal{K}}[N_A(I \otimes \rho_0)], &(82)\\
d_B &= \mathrm{Tr}_{\mathcal{K}}[D_B(I \otimes \rho_0)]. &(83)
\end{aligned}$$

The noise operator N_A is said to be *statistically independent of the object* **S** if n_A is scalar, and moreover the disturbance operator D_B is *statistically independent of the object system* **S** if d_B is scalar. Then, we have the following characterizations of measurements that obey the Heisenberg-type noise-disturbance relation [17].

Theorem 8. *For any measuring process $\mathcal{M}$ and observables A, B, we have*

$$\epsilon(A)\eta(B) + \frac{1}{2}|\langle [n_A, B]\rangle - \langle [d_B, A]\rangle| \ge \frac{1}{2}|\langle [A, B]\rangle|. \tag{84}$$

Theorem 9. *If the noise and disturbance are statistically independent of the object system, we have the Heisenberg-type noise-disturbance relation.*

For measurement operators $\{M_m\}$, we have

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{K}}[N_A(I \otimes |\xi\rangle\langle\xi|)] &= \langle\xi|N_A|\xi\rangle_{\mathcal{K}} \\
&= \langle\xi|U^\dagger(I \otimes M)U - A \otimes I|\xi\rangle_{\mathcal{K}} \\
&= \langle\xi|U^\dagger(I \otimes M)U|\xi\rangle_{\mathcal{K}} - A \\
&= \sum_m m M_m^\dagger M_m - A.
\end{aligned}$$

Thus, we have

$$n_A = \sum_m m M_m^\dagger M_m - A. \tag{85}$$

On the other hand, we have

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{K}}[D_B(I \otimes \rho_0)] &= \langle\xi|D_B|\xi\rangle_{\mathcal{K}} \\
&= \langle\xi|U^\dagger(B \otimes I)U - B \otimes I|\xi\rangle_{\mathcal{K}} \\
&= \langle\xi|U^\dagger(B \otimes I)U|\xi\rangle_{\mathcal{K}} - B \\
&= T(B) - B \\
&= \sum_m M_m^\dagger B M_m - B.
\end{aligned}$$

Thus, we have

$$d_B = \sum_m M_m^\dagger B M_m - B. \tag{86}$$

Now, from Theorem 4 we have the following criterion for measurements satisfying the Heisenberg-type noise-disturbance uncertainty relation.

Theorem 10. *A measurement with measurement operators $\{M_m\}$ satisfies the Heisenberg-type noise-disturbance relation*

$$\epsilon(A)\eta(B) \ge \frac{1}{2}|\langle [A, B]\rangle|$$

if we have

$$[\sum_m m M_m^\dagger M_m - A, B] = [\sum_m M_m^\dagger B M_m - B, A]. \quad (87)$$

C. Typical violations of the Heisenberg-type noise-disturbance relation

If the Heisenberg-type noise-disturbance relation were to hold for bounded observables A, B with $\langle [A,B]\rangle \neq 0$, we would have no precise measurements with $\epsilon(A) = 0$ nor non-disturbing measurements with $\eta(B) = 0$. From the universal uncertainty principle, we have correct limitations on the noiseless or non-disturbing measurements [15].

The *uncertainty principle for non-disturbing measurements*, i.e., $\eta(B) = 0$, is given by

$$\epsilon(A)\sigma(B) \geq \frac{1}{2}|\langle [A,B]\rangle|. \quad (88)$$

The *uncertainty principle for noiseless measurements*, i.e., $\epsilon(A) = 0$, is given by

$$\sigma(A)\eta(B) \geq \frac{1}{2}|\langle [A,B]\rangle|. \quad (89)$$

From the above, we have the following statements.

Theorem 11. *For any measurement with measurement operators $\{M_m\}$, the relation*

$$\epsilon(A)\sigma(B) \geq \frac{1}{2}|\langle [A,B]\rangle| \quad (90)$$

holds if it satisfies

$$[M_m, B]\psi = 0 \quad (91)$$

for all m, and the relation

$$\sigma(A)\eta(B) \geq \frac{1}{2}|\langle [A,B]\rangle| \quad (92)$$

holds if it satisfies

$$m M_m\psi = M_m A\psi \quad (93)$$

for all m.

The assertions can be verified immediately, since $\eta(B) = 0$ follows from Eq. (91) and $\epsilon(A) = 0$ follows from Eq. (93).

D. Projective measurements of Pauli operators

In order to figure out the noise-disturbance relation for the qubit measurements, let X, Y, Z be the Pauli operators on the 2 dimensional state space $\mathbf{C}^2$, and consider the projective measurement of Z. In this case, the measurement operators are given by $M_{-1} = (I - Z)/2$, $M_1 = (I + Z)/2$, and $M_m = 0$ if $m \neq \pm 1$. Let ψ be an arbitrary state vector. Then, from Eq. (77) we have

$$\epsilon(Z) = 0. \quad (94)$$

On the other hand, we have

$$\begin{aligned}\eta(X)^2 &= \sum_{m=\pm 1} \|[M_m, X]\psi\|^2 \\ &= \|[\frac{I+Z}{2}, X]\psi\|^2 + \|[\frac{I-Z}{2}, X]\psi\|^2 \\ &= 2\|Y\psi\|^2,\end{aligned}$$

and since $\|Y\psi\| = 1$, we have

$$\eta(X) = \sqrt{2}. \quad (95)$$

We actually have $\eta(X) = \sqrt{2} \leq 2 = 2\|X\|$ as Eq. (78), and we have $\epsilon(Z)\eta(X) = 0$. Thus, the Heisenberg-type noise-disturbance relation is violated in the state with $\langle [X,Z]\rangle \neq 0$. On the other hand, the universal uncertainty relation holds, as we have

$$\begin{aligned}&\epsilon(Z)\eta(X) + \epsilon(Z)\sigma(X) + \sigma(Z)\eta(X) \\ &\quad = \sigma(Z)\eta(X) = \sqrt{2}\sigma(Z) \geq \sigma(X)\sigma(Z) \\ &\quad \geq \frac{1}{2}|\langle [Z,X]\rangle|.\end{aligned}$$

In particular, we have

$$\begin{aligned}(-1)M_{-1} &= (-1)\frac{I-Z}{2} = \frac{I-Z}{2}Z = M_{-1}Z, \quad (96)\\ M_1 &= \frac{I+Z}{2} = \frac{Z+I}{2}Z = M_1 Z, \quad (97)\end{aligned}$$

and

$$\sigma(Z)\eta(X) \geq \frac{1}{2}|\langle [Z,X]\rangle|. \quad (98)$$

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